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Least Square Approach to Estimate 3D Coordinate Transformation Parameters: A Case of Three Reference Systems in Sweden

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Abstract: Seven parameters of Helmert transformation are estimated using three-dimensional Cartesian coordinates in Sweden. Here, two cases are studied. Cartesian coordinates of RT90 and SWEREFF 93 in mm level accuracy for Case A are generated from secondary source. However for Case B, coordinates have been obtained by a field measurement. Trimble differential GPS has been used to measure coordinate in both SWEREFF 99 TM and RT90 reference systems in Gothenburg region, Sweden. It provides the coordinates in decimeter level accuracy. Helmert transformation parameters are estimated by applying MATLAB code. Seven parameters of Helmert transformation between RT90 and SWEREFF 93, and RT90 and SWEREFF 99 TM, and vice-versa are estimated. The average variance-covariance and, difference between measured and transformed coordinates in Case A is estimated to 3.86e-7 and 0.082 meter, respectively. However in Case B, the estimated transformation parameters is poor due to low level accuracy of measured coordinates and not fit in proper Cartesian system since the height component in 3D coordinate system provides geoid height which does not correspond to Cartesian coordinate. Therefore it gives high average variance-covariance as to 0.2165 and difference between measured and transformed coordinates to 5.498 meter. So estimation of Helmert transformation parameters requires Cartesian coordinates with high accuracy.

Key words: Cartesian coordinate, Helmert transformation parameters, Bursa-Wolf mathematical model, SWEREFF 93, SWEREFF 99 TM, RT90.

1. Introduction

All spatial data are idealized; a generalization or simplification of real world features (Heywood et al., 2011). To build meaningful spatial data infrastructure, it requires defining coordinate system that describes the position in the real world location. However, very often people apply different type of coordinate systems depending on scale and purpose, e.g., SWEREFF 99 TM projected coordinate system is used for official Swedish map production since 2007 (Lantmäteriet, N/A). Better comparison of different spatial data in different coordinate systems involves conversion into one coordinate system. This conversion is widely used in cartography, geodesy, photogrammetry, remote sensing related professionals. Conversion of one coordinate system to another requires transformation parameters.

Estimation transformation parameters are a mathematical operation which takes coordinate of one point in both two coordinate systems. Now the days many types of mathematical models are applied to estimate coordinate transformation parameters. Application of mathematical modules is also varied depending on two dimensional (2D) or three dimensional coordinate transformations.

The Bursa-Wolfs' (Bursa, 1962; Wolf, 1963) is such a mathematical model which is able to consider position, size and shape of network. The aim of this paper is to apply Bursa-Wolfs mathematical model to estimate 3D affine transformation parameters.

2. Mathematical model of Bursa-Wolf

The Bursa-wolfs mathematical model describes functional relationship of three dimensional rectangular Cartesian coordinate in pair wise. This model is well known as seven parameters of Helmert transformation. The general form of this mathematical model can be written as,

$$\begin{bmatrix} X_j \\ Y_j \\ Z_j \end{bmatrix}_a = \begin{bmatrix} \delta_X \\ \delta_Y \\ \delta_Z \end{bmatrix} + SR(\theta_X, \theta_Y, \theta_Z) \begin{bmatrix} X_j \\ Y_j \\ Z_j \end{bmatrix}_b , \quad 1 \le j \le n \qquad \text{Eq.1}$$

where, n is number of points those have known coordinates in two systems a and b; X, Y, and Z are 3D coordinate; δ_X, δ_Y and δ_Z are three translation parameters along three axis; θ_X, θ_Y and θ_Z, are three rotational angles around the X, Y and Z axis, respectively; S is the scale factor between both systems (Kutoglu et al., 2002; Al Marzooqi et al., 2005; Fan, 2005; Andrei, 2006; Guo, 2007). The R matrix in Eq. 1 can be written as,

$$R = R(\theta_X).R(\theta_Y).R(\theta_Z) = \begin{bmatrix} \cos\theta_Z & \sin\theta_Z & 0 \\ -\sin\theta_Z & \cos\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_Y & 0 & -\sin\theta_Y \\ 0 & 1 & 0 \\ \sin\theta_Y & 0 & \cos\theta_Y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_X & \sin\theta_X \\ 0 & -\sin\theta_X & \cos\theta_X \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_Y \cos\theta_Z & \cos\theta_X \sin\theta_Z + \sin\theta_X \sin\theta_Y \cos\theta_Z & \sin\theta_X \sin\theta_Z - \cos\theta_X \sin\theta_Y \cos\theta_Z \\ -\cos\theta_Y \sin\theta_Z & \cos\theta_X \cos\theta_Z - \sin\theta_X \sin\theta_Y \sin\theta_Z & \sin\theta_X \cos\theta_Z + \cos\theta_X \sin\theta_Y \sin\theta_Z \\ \sin\theta_Y & -\sin\theta_X \cos\theta_Y & \cos\theta_X \cos\theta_Y \end{bmatrix}. \qquad \text{Eq.2}$$

3. Computation

Helmert transformation seven parameters are based on least squares estimation. Since Eq.1 involves multidimensional point data, therefore it requires linearizing matrix system. Linearization can be described by general form of linear equation as

$$y = a_1 x_1 + \cdots + a_n x_n \qquad \text{Eq.3}$$

where $a_1 \dots a_n \in R^n$, y and x is variables where y depends to interdependent variable x. To make simplification, from second order terms in Eq.3 are neglected and thus Eq.3 becomes

$$y = Ax \qquad \text{Eq.4}$$

where $A \in R^n$ which denotes to design matrix (see Guo, 2007 for more details). The Eq.1 and Eq.4 are comparable and both have similar properties. The Eq.1 can be written as linear form as

$$X_a = \delta X + SRX_b \qquad \text{Eq.5}$$

3.1 Linearizing rotation matrix

Let, we initialize $\theta°_X$, $\theta°_Y$, $\theta°_Z = 0$ and calculate initial rotation matrix $R°$ by applying a direct method as (Fan, 2005)

$$R° = \begin{bmatrix} r°_{11} & r°_{12} & r°_{13} \\ r°_{21} & r°_{22} & r°_{23} \\ r°_{31} & r°_{32} & r°_{33} \end{bmatrix} \qquad \text{Eq.6}$$

where all elements r_{ij} (i,j = 1,2,3) in R matrix are function of θ_X, θ_Y and θ_Z. It gives approximate rotation vector which requires correction. It can be corrected as

$$\left.\begin{aligned} \theta_X &= \theta°_X + \delta\theta_X \\ \theta_Y &= \theta°_Y + \delta\theta_Y \\ \theta_Z &= \theta°_Z + \delta\theta_Z \end{aligned}\right\} \qquad \text{Eq.7}$$

where $\delta\theta_x$, $\delta\theta_y$ and $\delta\theta_z$ are corresponding correction. The Eq.7 can be corrected and linearized as

$$r_{ij} = r_{ij}(\theta_X, \theta_Y, \theta_Z) \qquad \text{Eq.8}$$

$$= r_{ij}(\theta°_X + \delta\theta_X;\ \theta°_Y + \delta\theta_Y;\ \theta°_Z + \delta\theta_Z)$$

$$\approx r_{ij}(\theta°_X, \theta°_Y, \theta°_Z) + \frac{\delta r_{ij}}{\delta\theta_X}\delta\theta_X + \frac{\delta r_{ij}}{\delta\theta_Y}\delta\theta_Y + \frac{\delta r_{ij}}{\delta\theta_Z}\delta\theta_Z$$

$= r^{o}_{ij} + e_{ij}\delta\theta_X + f_{ij}\delta\theta_Y + g_{ij}\delta\theta_Z$

where e_{ij}, f_{ij} and g_{ij} is 3X3 matrix, and

$$e_{ij} = \frac{\delta r_{ij}}{\delta\theta_X} = \begin{bmatrix} 0 & -r^0_{13} & r^0_{12} \\ 0 & -r^0_{23} & r^0_{22} \\ 0 & -r^0_{33} & -r^0_{32} \end{bmatrix}, \qquad e_{ij} = \frac{\delta r_{ij}}{\delta\theta_Y} = \begin{bmatrix} -\sin\theta^0_Y\cos\theta^0_Z & -r^0_{32}\cos\theta^0_Z & -r^0_{33}\cos\theta^0_Z \\ \sin\theta^0_Y\sin\theta^0_Z & r^0_{32}\sin\theta^0_Z & r^0_{33}\sin\theta^0_Z \\ \cos\theta^0_Y & \sin\theta^0_X\sin\theta^0_Y & -\cos\theta^0_X\sin\theta^0_Y \end{bmatrix}$$ and

$$g_{ij} = \frac{\delta r_{ij}}{\delta\theta_Z} = \begin{bmatrix} r^0_{21} & r^0_{22} & r^0_{23} \\ -r^0_{11} & -r^0_{12} & -r^0_{13} \\ 0 & 0 & 0 \end{bmatrix}.$$

The Eq. 8 can be rewrite as linear form as

$$R = R(\theta_X, \theta_Y, \theta_Z) = R^0 + E\delta\theta_X + F\delta\theta_Y + G\delta\theta_Z. \qquad \text{Eq.9}$$

3.2 Linearizing scale matrix

For a single scale factor, the linearized scale matrix, *S* can be written as

$$S = S^o + \delta_S \qquad \text{Eq.10}$$

where S^o is initialized to 1 and δ_S is correction (e.g., Andrei, 2006; Fan, 2005). The δ_S is often expressed as part per million (ppm). But it can be parts per billion also (e.g., Altamimi et al, 2011).

3.3 Linearizing Bursa-Wolfs model matrix

The Bursa-Wolfs model, Eq. 1 can be linearized by inserting Eq. 9 and Eq. 10 into Eq. 5 in case of single scale factor as

$$X_a = \delta X + (S^o + \delta_S)(R^0 + E\delta\theta_X + F\delta\theta_Y + G\theta_Z)X_b \qquad \text{Eq.11}$$

$$\approx \delta X + (R^0 S^o + R^0\delta_S + E\delta\theta_X S^o + F\delta\theta_Y S^o + G\theta_Z S^o)X_b.$$

The Eq. 11 can be a simpler matrix form as

$$\underset{3\times3}{\mathbf{L}_j} = [l_1;\ \ l_2;\ \ l_3] = \underset{3\times7}{\mathbf{A}_j}\,\underset{7\times3}{\delta X} \qquad \text{Eq.12}$$

where $j = 1,2\ldots,n,$ number of observations, L_j is observation vector, A_j is design matrix and δX is unknown transformation parameters to be calculated, can be inserted following as

$$\underset{3\times3}{\mathbf{L}_j} = X_a - R^o S^o X_b,$$

$$\underset{3\times7}{\mathbf{A}_j} = \begin{bmatrix} 1 & 0 & 0 & a_{14} & a_{15} & a_{16} & a_{17} \\ 0 & 1 & 0 & a_{24} & a_{25} & a_{26} & a_{27} \\ 0 & 0 & 1 & a_{34} & a_{35} & a_{36} & a_{37} \end{bmatrix},$$

$$\underset{7\times3}{\delta\mathbf{X}} = [\delta X;\ \delta Y;\ \delta Z;\ \delta S;\ \delta\theta_X;\ \delta\theta_Y;\ \delta\theta_Z].$$

Elements of observation vector and design matrix can be obtain for single scale factor following as

$$l_1 = X^a_j - S^o(r^o_{11}X^b_j + r^o_{12}Y^b_j + r^o_{13}Z^b_j),$$
$$l_2 = Y^a_j - S^o(r^o_{21}X^b_j + r^o_{22}Y^b_j + r^o_{23}Z^b_j),$$
$$l_3 = Z^a_j - S^o(r^o_{31}X^b_j + r^o_{32}Y^b_j + r^o_{33}Z^b_j),$$

$a_{14} = r_{11}^o X_j^b + r_{12}^o Y_j^b + r_{13}^o Z_j^b,$
$a_{24} = r_{21}^o X_j^b + r_{22}^o Y_j^b + r_{23}^o Z_j^b,$
$a_{34} = r_{31}^o X_j^b + r_{32}^o Y_j^b + r_{33}^o Z_j^b,$
$a_{15} = S^o(-r_{13}^o Y_j^b + r_{12}^o Z_j^b),$
$a_{25} = S^o(-r_{23}^o Y_j^b + r_{22}^o Z_j^b),$
$a_{35} = S^o(-r_{33}^o Y_j^b + r_{32}^o Z_j^b),$
$a_{16} = -S^o cos\theta_Z^o(sin\theta_Y^o X_j^b + r_{32}^o Y_j^b + r_{33}^o Z_j^b),$
$a_{26} = S^o sin\theta_Z^o(sin\,\theta_Y^o X_j^b + r_{32}^o Y_j^b + r_{33}^o Z_j^b),$
$a_{36} = S^o(cos\theta_Y^o X_j^b + sin\,\theta_X^o sin\theta_Y^o Y_j^b - cos\theta_X^o sin\theta_Y^o Z_j^b),$
$a_{17} = S^o(r_{21}^o X_j^b + r_{22}^o Y_j^b + r_{23}^o Z_j^b),$
$a_{27} = -S^o(r_{11}^o X_j^b + r_{12}^o Y_j^b + r_{13}^o Z_j^b),$
$a_{37} = 0,$

3.4 Estimation of transformation parameters

The computed transformation parameters obtained from Eq. 11 or Eq. 12 are not error free. These can be corrected by applying least square estimation as (Fan, 2005)

$$\delta\hat{X} = (A^T C^{-1} A)^{-1} A^T C^{-1} L \qquad \text{Eq.13}$$

where $\delta\hat{X}$ is corrected form of δX vector and $C = I$. The Eq. 13 gives seven transformation parameters following as

$$\begin{aligned} &\delta_X, \delta_Y, \delta_Z = \delta\hat{X}(1, 2,3), \\ &\hat{S} = S^o + \delta\hat{S} = \delta\hat{X}(4), \\ &\hat{\theta}_X = \theta_X^o + \delta\theta_X = \delta\hat{X}(5), \\ &\hat{\theta}_Y = \theta_Y^o + \delta\theta_Y = \delta\hat{X}(6), \\ &\hat{\theta}_Z = \theta_Z^o + \delta\theta_Z = \delta\hat{X}(7) \end{aligned} \qquad \text{Eq.14}$$

3.5 Validation

Calculation of transformation parameters is based on least square estimation. This process generates residual. This residual can be estimated using Eq. 12 or Eq. 13 as

$$\varepsilon = L - A\delta\hat{X} \qquad \text{Eq.15}$$

where $\delta\hat{X}$ is replaced with δX. A variance-covariance matrix (VC_m) of transformation parameters can be computed based on so called a posteriori error estimation of variance factors ($\hat{\sigma}_o^2$) which is minimized with increasing observation and number of parameters. It can be written as

$$VC_m = \hat{\sigma}_o^2 (A^T C^{-1} A)^{-1} \qquad \text{Eq.16}$$

where $\hat{\sigma}_o^2 = \frac{\varepsilon^T C^{-1} \varepsilon}{O_p j - P_n}$, O_p is number of observation parameters, j is number of observations and P_n is number of transformation parameters is being estimated.

4. Case study

To calculate seven parameters of Helmert transformation, Matlab code is applied (Appendix A). This code is rather general. It should be able to calculate Helmert seven transformation parameters of given any two Cartesian coordinate systems. Here two examples (Cases A and B) are presented where Case A is taken to examine the transformation system with Matlab code and Case B is given from new coordinate measurement in two systems.

4.1 Case A

The Cartesian geocentric coordinates (Tab. 1) was obtained in autumn 2005 from Division of Geodesy, Royal Institute of Technology, Sweden. These data was also used by Andrei (2006) but all coordinates are not probably arranged in proper order. Therefore it is very difficult to make relation of sites' coordinate in direction to easting, northing and height components. However, these 20 common sites coordinates in coordinate systems *a* and *b* are presented in table 1. Here, *a* and *b* is indicated to coordinate system of RT90 (Rikstrianguleringen) and SWEREFF 93, respectively in Sweden. The SWEREFF 93 is transformed from ETRF89 which was combined from EUREFF 89 GPS field campaign, DOSE and the International Terrestrial Reference Frame 91. The unit of all parameters of these two coordinate systems is meter.

Table 1: Cartesian geocentric coordinates of common sites

Site Id	RT90 (in m)			SWEREFF 93 (in m)		
	X_b	Y_b	Z_b	X_a	Y_a	Z_a
1	2441775.419	799268.100	5818729.162	2441276.712	799286.666	5818162.025
2	3464655.838	845749.989	5270271.528	3464161.275	845805.461	5269712.429
3	3309991.828	828932.118	5370882.280	3309496.800	828981.942	5370322.060
4	3160763.338	759160.187	5469345.504	3160269.913	759204.574	5468784.081
5	2248123.493	865686.595	5886425.596	2247621.426	865698.413	5885856.498
6	3022573.157	802945.690	5540683.951	3022077.340	802985.055	5540121.276
7	3104219.427	998384.028	5463290.505	3103716.966	998426.412	5462727.814
8	2998189.685	931451.634	5533398.462	2997689.029	931490.201	5532835.154
9	3199093.294	932231.327	5420322.483	3198593.776	932277.179	5419760.966
10	3370658.823	711876.990	5349786.786	3370168.626	711928.884	5349227.574
11	3341340.173	957912.343	5330003.236	3340840.578	957963.383	5329442.724
12	2534031.166	975174.455	5752078.309	2533526.497	975196.347	5751510.935
13	2838909.903	903822.098	5620660.184	2838409.359	903854.897	5620095.593
14	2902495.079	761455.843	5609859.672	2902000.172	761490.908	5609296.343
15	2682407.890	950395.934	5688993.082	2681904.794	950423.098	5688426.909
16	2620258.868	779138.041	5743799.267	2619761.810	779162.964	5743233.630
17	3246470.535	1077900.355	5365277.896	3245966.134	1077947.976	5364716.214
18	3249408.275	692757.965	5426396.948	3248918.041	692805.543	5425836.841
19	2763885.496	733247.387	5682653.347	2763390.878	733277.458	5682089.111
20	2368885.005	994492.233	5818478.154	2368378.937	994508.273	5817909.286

Table 2 shows the seven parameters of Helmert transformation from coordinate systems RT90 to SWEREFF 93 with their variance-covariance which are similar with Andrei (2006). This result is obtained from adjustment model where all coordinates have similar weight that is described according to Reit (1999). The average variance-covariance is obtained to 3.86e-7. Note that inverse transformation from SWEREFF 93 to RT90 is followed by changing the sigh of the transformation parameters.

Table 2: Seven parameters of Helmert transformation

Parameters	Value	Variance-Covariance
δ_X (m)	-419.5684	3.9396e-007
δ_Y (m)	-99.2460	1.4370e-006
δ_Z (m)	-591.4559	4.2571e-007
δ_S (ppm)	1.0237	5.9663e-008
$\delta\theta_X$ (arsec)	0.8502	2.0535e-007
$\delta\theta_Y$ (arsec)	1.8141	6.2006e-008
$\delta\theta_Z$ (arsec)	-7.8535	1.1634e-007

4.2 Case B

A filed work has been conducted to collect coordinates of some common sites in Gothenburg region (near 57.689195° latitude, 11.966780° longitude), Sweden in autumn, 2013. There are 10 common sites' coordinates in both SWEREF 99 TM and RT90 coordinate systems collected using Trimble differential GPS. The coordinates of two different coordinate systems are presented in table 3. All sites are measured two times in each coordinate system to check their consistency. They give decimeter level accuracy. Here easting and northing components are in Cartesian system but height component gives height from geoid. The Swedish SWEN05LR Geoid model is used as reference for height component. So, third component (height) of both coordinate systems does not correspond to Cartesian system.

Table 3: Coordinates (in m) of ten common sites in two different coordinate systems

Site Id	SWEREF 99TM (m)			RT90 (m)		
	Easting	Northing	Geoid height	Easting	Northing	Geoid height
1	319163.2	6398188	63.408	1271102	6402555	65.443
2	319165.1	6398184	67.314	1271103	6402549	64.45
3	319149.6	6398191	73.83	1271085	6402553	65.958
4	319162.6	6398173	63.742	1271100	6402541	63.297
5	319179.8	6398162	62.934	1271119	6402529	66.967
6	319189.6	6398159	70.388	1271127	6402524	64.711
7	319203.4	6398147	61.847	1271143	6402512	66.617
8	319243.7	6398123	62.437	1271181	6402489	64.109
9	319237.4	6398147	73.744	1271181	6402510	60.898
10	319245.1	6398081	63.62	1271182	6402447	63.616

Table 4 shows the seven parameters of Helmert transformation from coordinate system SWEREFF 99 TM to RT90 with their variance-covariance. The average variance-covariance is calculated to 0.2165. Similar as Case A, inverse transformation can be performed by changing sign of transformation parameters.

Table 4: Seven parameters of Helmert transformation

7- Parameters	Value	Variance-Covariance
δ_X (m)	1027871.7679	0.2535
δ_Y (m)	72364.1120	0.2498
δ_Z (m)	941534.6321	0.7243
δ_S (ppm)	-266.6595	0.0359
$\delta\theta_X$ (arsec)	29421.7784	0.1122
$\delta\theta_Y$ (arsec)	-25330.9216	0.1020
$\delta\theta_Z$ (arsec)	1235.9212	0.0378

4.3 Comparisons

These transformation parameters applied to calculate the coordinates from one system to another. In Case A, the transformed coordinates are fairly similar with measured coordinates. It provides coordinates in decimeter-millimeter difference between two systems (Tab. 5). The average difference between measured and transformed coordinates is 0.082 meter. However, in Case B, transformed coordinates have accuracy decameter to decimeter level compare to measured coordinates (Tab. 5). Easting and northing components have accuracy higher than height component. The average difference between measured and transformed coordinates in Case B is 5.498 meter. This curse accuracy in Case B, because of low accuracy of measured coordinates in both SWEREFF 99 TM and RT90 coordinates systems. Second problem was height component which was not belongs to Cartesian system.

Table 5: Difference between measured and transformed coordinates

Site Id	Case A (in m)			Case B (in m)		
	X_b	Y_b	Z_b	Easting	Northing	Height
1	0.026	0.042	0.181	2.978	6.074	14.625
2	0.017	0.215	0.024	1.570	3.492	9.464
3	0.045	0.059	0.043	1.766	0.449	3.662
4	0.080	0.031	0.236	1.360	6.868	9.961
5	0.064	0.321	0.128	3.267	5.971	14.980
6	0.036	0.130	0.165	0.599	2.944	6.180
7	0.003	0.101	0.015	3.812	4.110	16.472
8	0.021	0.101	0.047	1.479	5.018	14.935
9	0.054	0.000	0.096	6.625	0.630	3.232
10	0.082	0.026	0.056	0.464	4.425	7.527
11	0.093	0.118	0.049			
12	0.029	0.049	0.074			
13	0.031	0.136	0.132			
14	0.026	0.101	0.005			
15	0.048	0.141	0.215			
16	0.015	0.067	0.105			
17	0.077	0.043	0.171			
18	0.115	0.030	0.071			
19	0.023	0.088	0.116			
20	0.118	0.093	0.104			

5. Conclusions

The traditional technique of estimating coordinate transformation parameters is based on linearized mathematical model. The seven parameters of Helmert transformation is easier to estimate and eventually give precise transformation parameters those are calculated from least square sense. It is in fact the most application method to estimate transformation parameters and to transform from one coordinate system to another. However, in Case A where the coordinates of common sites' were Cartesian system with mm level accuracy provides precise transformation parameters as average variance-covariance as 3.86e-7. The average difference between measured and transformed coordinates is obtained to cm level as to 8.2 cm. On the other hand in Case B, the coordinates were not perfect Cartesian system and accuracy were decimeter level. In this case the average variance-covariance as high as 0.2165 and the average difference between measured and transformed coordinates is meter level as to 5.5 m. Therefore, general ideas of estimating transformation parameters using this method are the pair coordinates of the common sites' have to be Cartesian and accurate enough to propagate minimum level of error.

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Reference

Al Marzooqi, Y., Fashir, H. and Ahmed, S.I. (2005). Derivation of Datum Transformation Parameters for Dubai Emirate. From Pharaohs to Geoinformatics, FIG Working Week 2005 and GSDI-8, 1-17.

Altamimi, Z., Collilieux, X. and Metivier, L. (2011). ITRF2008: an improved solution of the international terrestrial reference frame. J. Geod, 85, 457-473.

Andrei, C.O. (2006). 3D affine coordinate transformations. TRITA Master's Thesis, Division of Geodesy, Royal Institute of Technology (KTH), Stockholm, Sweden, 06-004(GIT), 1-50.

Bursa, M. (1962). The theory of the determination of the non-parallelism of the minor axis of the reference ellipsoid and the inertial polar axis of the Earth, and the planes of the initial astronomic and geodetic meridians form observations of artificial Earth Satellite. Studia Geophysica et Geodetica, 6, 209-214.

Fan, H. (2005). Theoretical Geodesy. Department of Geodesy, Report no. 2016, KTH, Stockholm, Sweden.

Guo, R. (2007). Systematical analysis of the transformation procedures in Baden-Württemberg with Least Squares and Total Least Squires methods. Geodetic Institute, University of Stuttgart, Stuttgart, Germany.

Heywood, I., Cornelius, S. and Carver, S. (2011). An Introduction to Geographical Information Systems. 4th ed., Prentice Hall, UK.

Kutoglu, H.S., Mekik, C. and Akcin, H. (2002). A comparison of Two Well Known Models for 7-Parameters Transformation. The Australian Surveyor, 1(47), 24-30.

Lantmäteriet. (N/A). RT 90 –SWEREFF 99. Available from: http://www.lantmateriet.se/en/Maps-and-geographic-information/GPS-and-geodetic-surveys/Geodesy/Transformations/RT-90---SWEREF-99/ (September 16, 2013).

Reit B.G. (1999). Improving a horizontal datum without changing the coordinates. Survey Review, 35(272), 400-404.

Wulf, H. (1963). Geometric connection and re-orientation of three-dimensional triangulation nets. Bulletin Geodesique, 68, 165-169.

Appendix A:

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%*************************************************************************
% Matlab code to Estimation of Helmert Transformation Parameters
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% Date: Sep 16, 2013
%#########################################################################

format long g;     clc;       clear;

load data.txt;
data= data/1000000;                % data (in m) are divided by one million (to make ppm)

S1=1;   A1=0;   A2=0;   A3=0;      % initialization of scale factor and other three rotation angles

p=206265;                          % radian to arcsec
C1=eye(60,60);
C=inv(C1);

del_S=1;                           % initialization of difference of scale factor and rotation angles
del_A1=1;
del_A2=1;
del_A3=1;
while (abs(del_S)>1e-10)&(abs(del_A1)>1e-10)&(abs(del_A2)>1e-10)&(abs(del_A3)>1e-10)

A=[];
L=[];
```

```
 R=[cos(A2)*cos(A3), cos(A1)*sin(A3)+sin(A1)*sin(A2)*cos(A3),    sin(A1)*sin(A3)-
cos(A1)*sin(A2)*cos(A3);
   -cos(A2)*sin(A3), cos(A1)*cos(A3)-sin(A1)*sin(A2)*sin(A3),    sin(A1)*cos(A3)+cos(A1)*sin(A2)*sin(A3);
    sin(A2),       -sin(A1)*cos(A2),                    cos(A1)*cos(A2)];
                                                                  % R matrix in section 2
r11o=R(1,1);       r12o=R(1,2);       r13o=R(1,3);                % Linearization of rotation matrix
r21o=R(2,1);       r22o=R(2,2);       r23o=R(2,3);
r31o=R(3,1);       r32o=R(3,2);       r33o=R(3,3);

    for i=1:size(data(:,1))
    a14=(r11o*data(i,2)+r12o* data(i,3)+r13o*data(i,4));
    a24=(r21o*data(i,2)+r22o* data (i,3)+r23o*data(i,4));
    a34=(r31o*data(i,2)+r32o*data(i,3)+r33o*data (i,4));
    a15=(S1*(-r13o*data(i,3)+r12o*data(i,4)));
    a25=(S1*(-r23o*data(i,3)+r22o*data(i,4)));
    a35=(S1*(-r33o*data(i,3)+r32o*data(i,4)));
    a16= -S1*cos(A3)*(sin(A2)*data(i,2)+r32o*data(i,3)+r33o*data(i,4));
    a26= S1*sin(A3)*(sin(A2)*data(i,2)+r32o*data(i,3)+r33o*data(i,4));
    a36= S1*(cos(A2)*data(i,2)+sin(A1)*sin(A2)*data(i,3)-cos(A1)*sin(A2)*data(i,4));
    a17=(S1*(r21o*data(i,2)+r22o*data(i,3)+r23o*data(i,4)));
    a27=(-S1*(r11o*data(i,2)+r12o*data(i,3)+r13o*data(i,4)));
    a37=0;

    Ai =[1,  0,  0,  a14,    a15,    a16,    a17;
         0,  1,  0,  a24,    a25,    a26,    a27;
         0,  0,  1,  a34,    a35,    a36,    a37;];                % Linearization of model matrix in section 3.3
   A=[A;Ai];

    L1=(data(i,5)-S1*(r11o*data(i,2)+r12o*data(i,3)+r13o*data(i,4)));
    L2=(data(i,6)-S1*(r21o*data(i,2)+r22o*data(i,3)+r23o*data(i,4)));
    L3=(data(i,7)-S1*(r31o*data(i,2)+r32o*data(i,3)+r33o*data(i,4)));

  Li=[L1;L2;L3];   L=[L;Li];
end;
delX = inv(A'*C*A)*A'*C*L;                 % Estimation of 7 transformation parameters, see section 3.4

       del_S = delX(4);
       del_A1 = delX(5);
       del_A2 = delX(6);
       del_A3 = delX(7);

       S1= S1+del_S;
       A1=A1+del_A1;
       A2= A2+del_A2;
       A3= A3+del_A3;

end
dx=delX(1)*1000000;      dy=delX(2)*1000000;      dz=delX(3)*1000000;      S1=(S1-1)*1000000;
                    % Conversion from ppm to m

A1=A1*p;           A2=A2*p;           A3=A3*p;                    % Conversion from radian to arcsec
disp(sprintf('dx = %.12f , \ndy = %.12f ,\ndz = %.12f',dx, dy, dz))
disp(sprintf('S1 = %.12f , \nA1 = %.12f ,\nA2 = %.12f,\nA3 =%.12f',S1,A1,A2,A3))

ep_cap=L-A*delX;                                    % Validation estimation
CO_v=(ep_cap'*C*ep_cap)/(3*20-7);                   % Variance-covariance matrix, see section 3.5
CXX=(CO_v)*inv(A'*C*A);
CXX=[sqrt(CXX(1,1)); sqrt(CXX(2,2)); sqrt(CXX(3,3)); sqrt(CXX(4,4)); sqrt(CXX(5,5)); sqrt(CXX(6,6));
sqrt(CXX(7,7)) ]

          % End, here you go!
```